Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

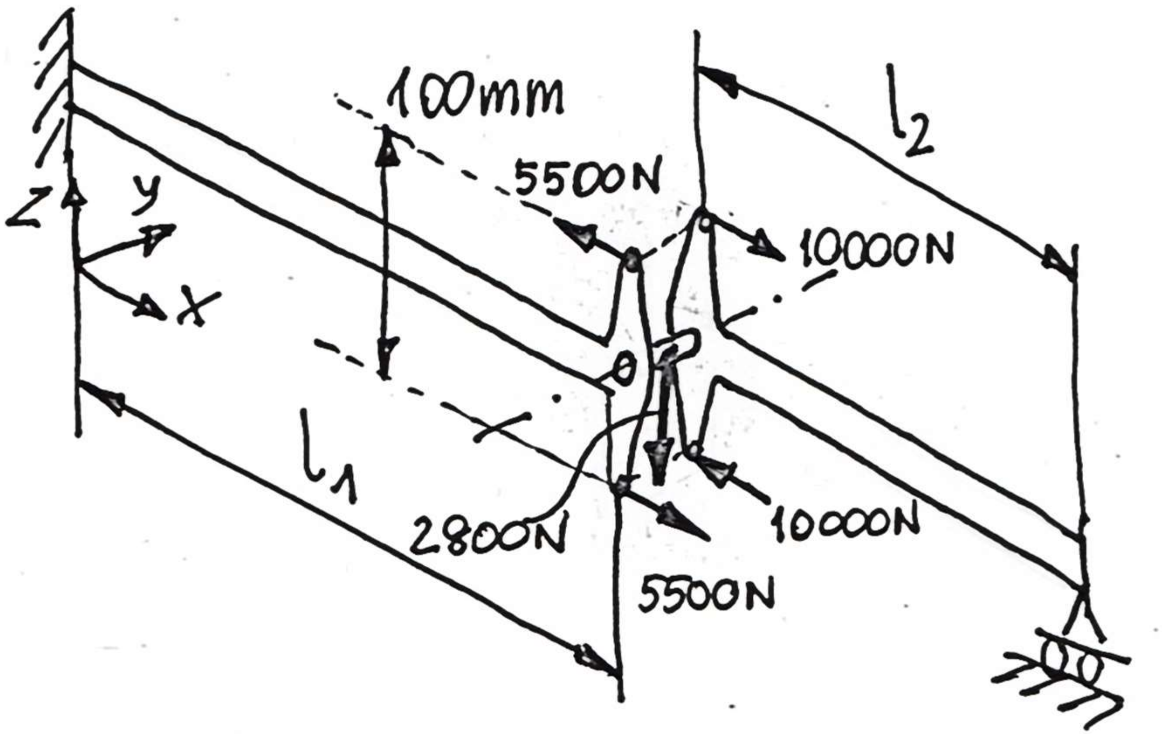
Finite element method (FEM)

Beam with a hinge

05.2021

EXAMPLE : BEAM WITH A HINGE

BUILD A FINITE ELEMENT MODEL. USE 2 FES.
FIND UNKNOWN DISPLACEMENTS, DEFLECTION, REACTIONS, INTERNAL FORCES AND CHECK THE EQUILIBRIUM.



$E = 2 \cdot 10^5$ MPa

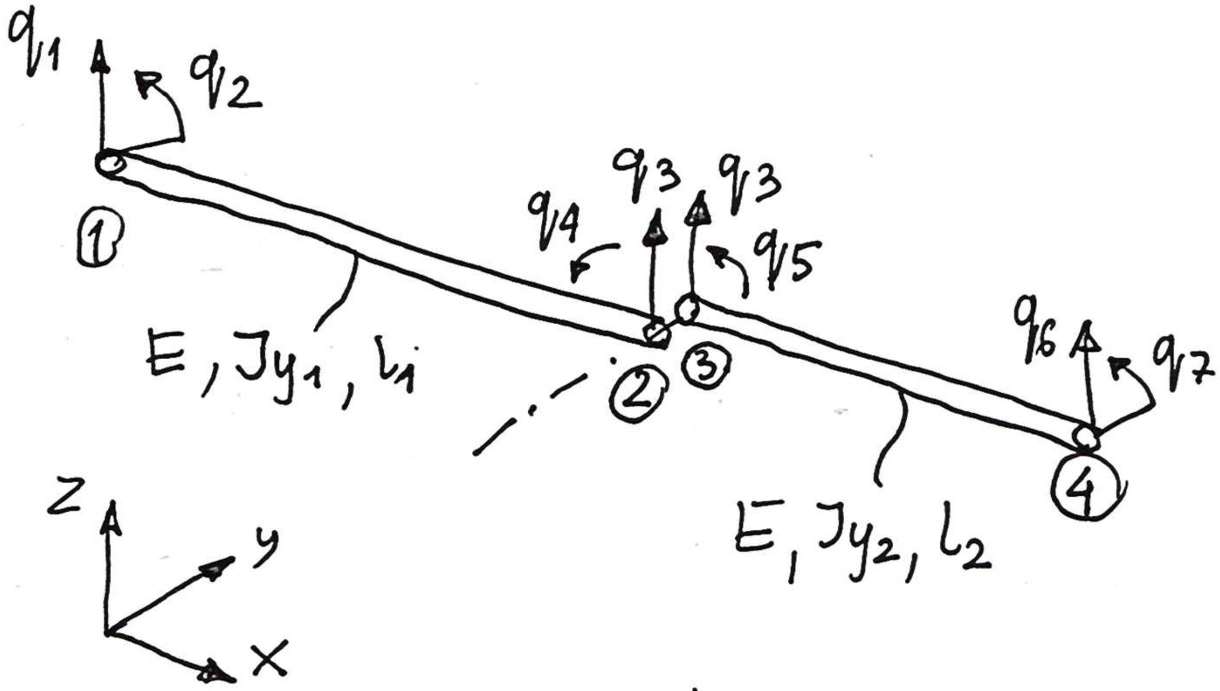
$l_1 = 1000$ mm

$l_2 = 500$ mm

$J_{y_1} = 1.143 \cdot 10^5$ mm^4

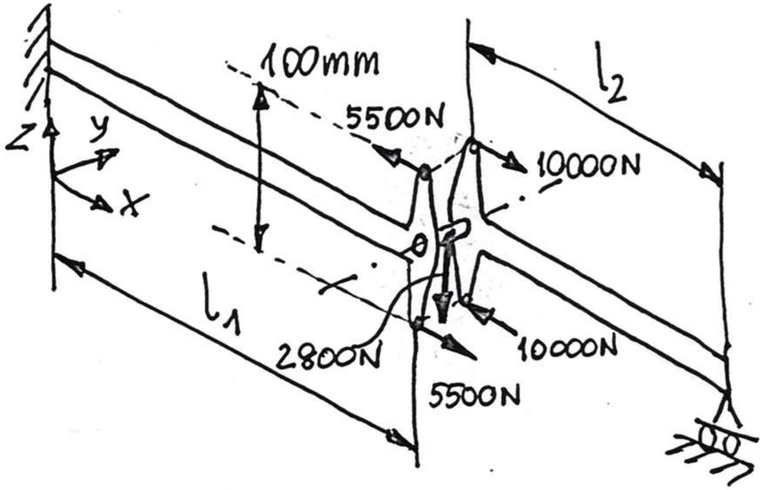
$J_{y_2} = 1.621 \cdot 10^5$ mm^4

NODAL PARAMETERS



$$\underset{1\times 7}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7 \rfloor$$

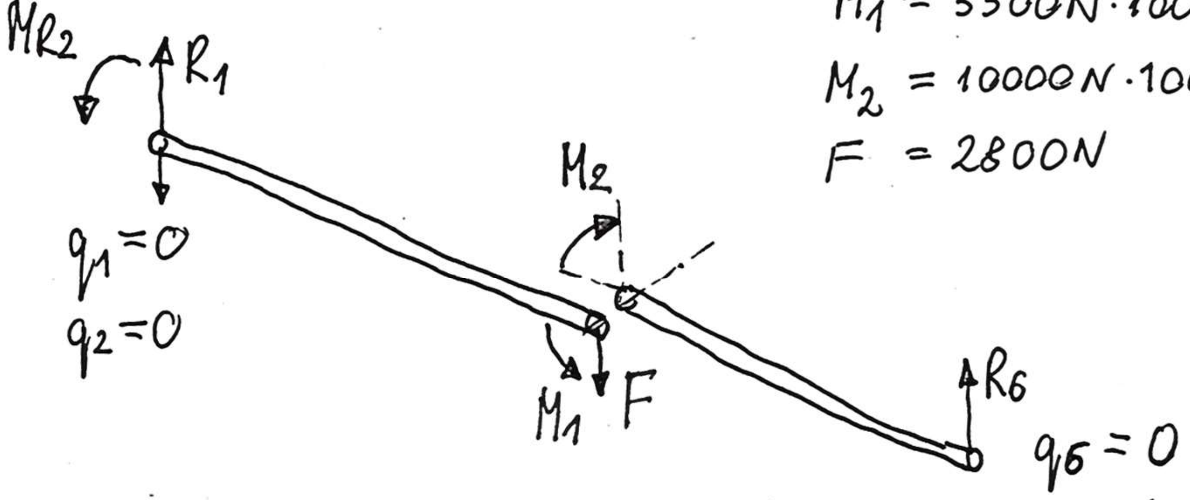
LOADS AND REACTIONS



$M_1 = 5500N \cdot 100mm = 0.55 \cdot 10^6\ Nmm$

$M_2 = 10000N \cdot 100\ mm = 1 \cdot 10^6\ Nmm$

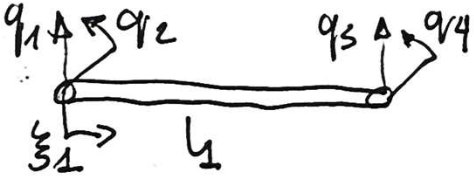
$F = 2800N$



$$\underset{1\times7}{\lfloor F \rfloor} = \lfloor R_1, M_{R2}, -F, M_1, -M_2, R_6, 0 \rfloor$$

STIFFNESS MATRICES

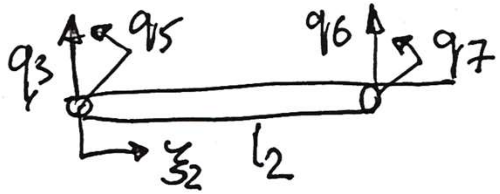
ELEMENT $\boxed{1}$



$$\underset{4\times4}{[k]_1} = \frac{2EJ_{y1}}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix} ;$$

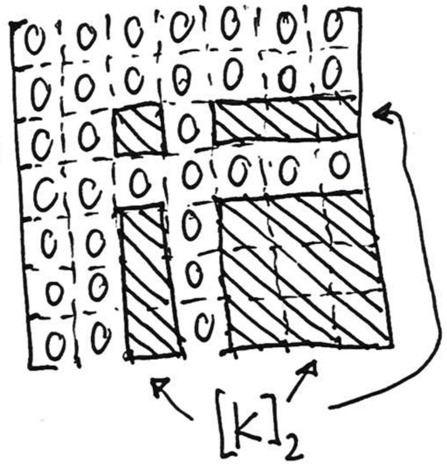
$$\underset{7\times7}{[k]^*_1} = \begin{bmatrix} [k]_1 & \underset{4\times3}{[0]} \\ \underset{3\times4}{[0]} & \underset{3\times3}{[0]} \end{bmatrix}$$

ELEMENT $\boxed{2}$



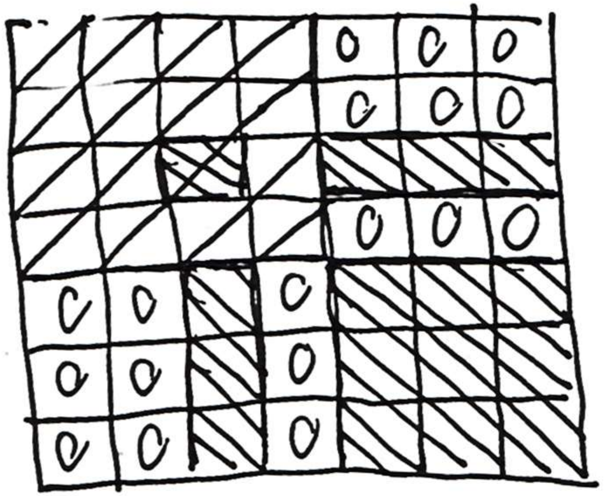
$$\underset{4\times4}{[k]_2} = \frac{2EJ_{y2}}{l_2^3}\begin{bmatrix} 6 & 3l_2 & -6 & 3l_2 \\ 3l_2 & 2l_2^2 & -3l_2 & l_2^2 \\ -6 & -3l_2 & 6 & -3l_2 \\ 3l_2 & l_2^2 & -3l_2 & 2l_2^2 \end{bmatrix} ;$$

$$\underset{7\times7}{[k]^*_2} =$$



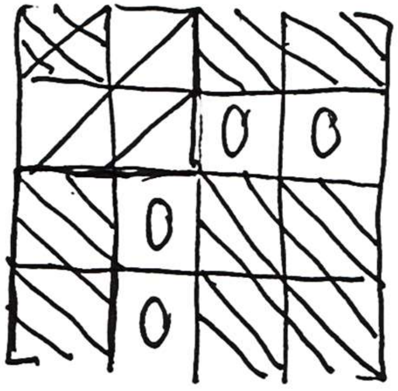
GLOBAL STIFFNESS MATRIX

$$\underset{7\times 7}{[K]} = [K]_1^* + [k]_2^* =$$

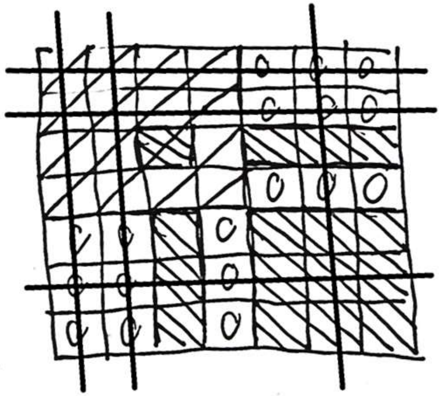


BOUNDARY CONDITIONS :

$$q_1 = 0\,,\quad q_2 = 0\,,\quad q_6 = 0 \quad \Rightarrow \quad N = NDOF - NOF = 4$$



$$\underbrace{\left[\;\right]}_{\substack{[K] \\ 4\times4}} \cdot \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_7 \end{Bmatrix} = \begin{Bmatrix} -F \\ M_1 \\ -M_2 \\ 0 \end{Bmatrix}$$



$$
\underset{4\times 4}{[K]} = \begin{bmatrix}
\frac{12EJy_1}{l_1^3}+\frac{12EJy_2}{l_2^3} & -\frac{6EJy_1}{l_1^2} & \frac{6EJy_2}{l_2^2} & \frac{6EJy_2}{l_2^2} \\
-\frac{6EJy_1}{l_1^2} & \frac{4EJy_1}{l_1} & 0 & 0 \\
\frac{6EJy_2}{l_2^2} & 0 & \frac{4EJy_2}{l_2} & \frac{2EJy_2}{l_2} \\
\frac{6EJy_2}{l_2^2} & 0 & \frac{2EJy_2}{l_2} & \frac{4EJy_2}{l_2}
\end{bmatrix}
$$

$$a = \frac{12EJy_1}{l_1^3}+\frac{12EJy_2}{l_2^3}$$

$$b = -\frac{6EJy_1}{l_1^2}, \quad c = \frac{6EJy_2}{l_2^2}$$

$$d = \frac{4EJy_1}{l_1}, \quad f = \frac{4EJy_2}{l_2}$$

$$
\underset{4\times 4}{[K]} = \begin{bmatrix}
a & b & c & c \\
b & d & 0 & 0 \\
c & 0 & f & f/2 \\
c & 0 & f/2 & f
\end{bmatrix}
$$

$$
\begin{cases}
a q_3 + b q_4 + c q_5 + c q_7 = -F \\
b q_3 + d q_4 = M_1 \\
c q_3 + f q_5 + \frac{f}{2} q_7 = -M_2 \\
c q_3 + \frac{f}{2} q_5 + f \cdot q_7 = 0
\end{cases}
$$

$$\overline{\text{III}} - \overline{\text{IV}}: \quad \frac{f}{2} q_5 - \frac{f}{2} q_7 = -M_2 \quad \Rightarrow \quad q_5 = q_7 - \frac{2M_2}{f}$$

$$\overline{\text{III}}: \quad c q_3 + f\left(q_7 - \frac{2M_2}{f}\right) + \frac{f}{2} q_7 = -M_2$$

$$c q_3 + \frac{3}{2} f \cdot q_7 = M_2 \quad \Rightarrow \quad q_7 = \frac{2(M_2 - c q_3)}{3f}$$

$$\overline{\text{II}}: \quad q_4 = \frac{M_1 - b q_3}{d}$$

$$q_5 = \frac{2(M_2 - c q_3)}{3f} - \frac{2M_2}{f} = -\frac{4M_2 + 2 c q_3}{3f}$$

I:

$$a \cdot q_3 + b\left(\frac{M_1 - b q_3}{d}\right) - c \cdot \frac{4 M_2 + 2 c q_3}{3f} + c \cdot \frac{2(M_2 - c q_3)}{3f} = -F$$

$$a \cdot q_3 + \frac{b}{d} M_1 - \frac{b^2}{d} q_3 - \frac{4 M_2 c}{3f} - \frac{2c^2 q_3}{3f} + \frac{2 M_2 c}{3f} - \frac{2c^2 q_3}{3f} = -F$$

$$\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right) q_3 = -F - \frac{b}{d} M_1 + \frac{2c M_2}{3f}$$

$$q_3 = \frac{\frac{2c M_2}{3f} - F - \frac{b}{d} M_1}{\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right)} = \frac{\frac{2 \cdot 6 E J y_2 \cdot M_2 \cdot l_2}{3 \cdot l_2^2 \cdot 4 E J y_2} - F + \frac{6 E J y_1 \cdot M_1 l_1}{l_1^2 \cdot 4 \cdot E J y_1}}{\left(a - \frac{b^2}{d} - \frac{4c^2}{3f}\right)}$$

$$a - \frac{b^2}{d} - \frac{4c^2}{3f} = \frac{12 E J y_1}{l_1^3} + \frac{12 E J y_2}{l_2^3} - \frac{36 E^2 J y_1^2}{l_1^4} \cdot \frac{l_1}{4 E J y_1} - \frac{4 \cdot 36 E^2 J y_2^2}{3 \cdot l_2^4} \cdot \frac{l_2}{4 E J y_2} = \frac{3 E J y_1}{l_1^3}$$

$$q_3 = \frac{\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) l_1^3}{3 E J y_1}$$

UNKNOWN NODAL PARAMETERS

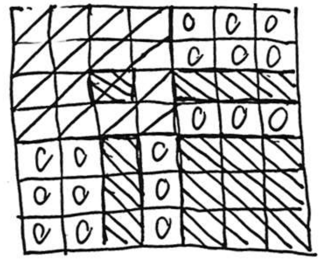
$$q_3 = \frac{\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) l_1^3}{3EJ_{y1}} = 0.3645\,mm.$$

$$q_4 = \frac{\left(M_1 + \frac{M_2 l_1}{2 l_2} - \frac{F l_1}{2}\right) l_1}{EJ_{y1}} = 6.5617 \cdot 10^{-3} = 0.38^\circ$$

$$q_5 = -\frac{M_2 l_2 + \frac{J_{y2} l_1^3}{J_{y1} l_2}\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3EJ_{y2}} = -5.8699 \cdot 10^{-3} = -0.34^\circ$$

$$q_7 = \frac{\frac{1}{2} M_2 l_2 - \frac{J_{y2} l_1^3}{J_{y1} l_2}\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3EJ_{y2}} = 1.8415 \cdot 10^{-3} = 0.11^\circ$$

REACTIONS:

$$[K]_{7\times 7} =$$ 

$$R_1 = \frac{-12\,EJ_{y1}}{l_1^{\,3}}\cdot q_3 + \frac{6\,EJ_{y1}}{l_1^{\,2}}\cdot q_4 =$$

$$= -4\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) + \frac{6}{l_1}\left(M_1 + \frac{M_2\, l_1}{2\, l_2} - \frac{F l_1}{2}\right) = F - \frac{M_2}{l_2} = 800N$$

$$M_{R2} = -\frac{6EJ_{y1}}{l_1^{\,2}}\cdot q_3 + \frac{2EJ_{y1}}{l_1}\, q_4$$

$$= -2l_1\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) + 2\left(M_1 + \frac{M_2\, l_1}{2\, l_2} - \frac{F l_1}{2}\right) =$$

$$= -\left(3M_1 + 2\frac{M_2\, l_1}{l_2} - 2Fl_1\right) + 2M_1 + M_2\frac{l_1}{l_2} - Fl_1 =$$

$$= Fl_1 - M_1 - M_2\frac{l_1}{l_2} = 0.25\cdot 10^{6}\, Nmm$$

$$R_6 = \frac{-12 E J_{y2}}{l_2^3} \cdot q_3 + 0 \cdot q_4 - \frac{6 E J_{y2}}{l_2^2} \cdot q_5 - \frac{6 E J_{y2}}{l_2^2} \cdot q_7 =$$

$$= -\frac{12 E J_{y2}}{l_2^3} \cdot \frac{\left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) l_1^3}{3 E J_{y1}} +$$

$$- \frac{6 E J_{y2}}{l_2^2} \cdot \left(- \frac{M_2 l_2 + \frac{J_{y2}}{J_{y1}} \frac{l_1^3}{l_2} \left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3 E J_{y2}} \right) +$$

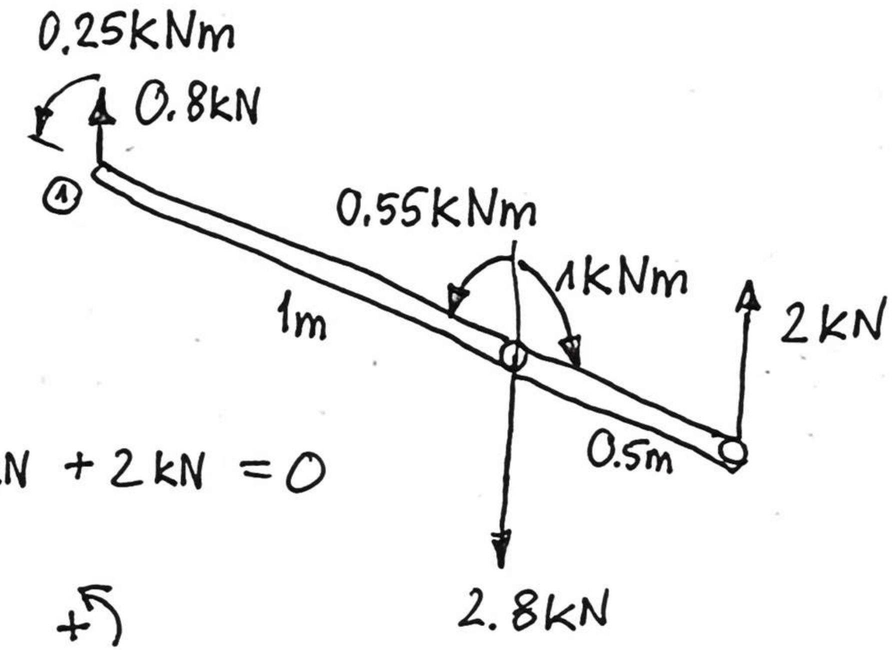
$$- \frac{6 E J_{y2}}{l_2^2} \cdot \left(\frac{\frac{1}{2} M_2 l_2 - \frac{J_{y2}}{J_{y1}} \frac{l_1^3}{l_2} \left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)}{3 E J_{y2}} \right) =$$

$$= -4 \frac{J_{y2}}{J_{y1}} \frac{l_1^3}{l_2^3} \left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right) +$$

$$+ 2 \left(\frac{M_2}{l_2} + \frac{J_{y2}}{J_{y1}} \frac{l_1^3}{l_2^3} \left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)\right) +$$

$$- 2 \left(\frac{1}{2}\frac{M_2}{l_2} - \frac{J_{y2}}{J_{y1}} \frac{l_1^3}{l_2^3} \left(\frac{3}{2}\frac{M_1}{l_1} + \frac{M_2}{l_2} - F\right)\right) = \frac{M_2}{l_2} = 2\,000 \text{ N}$$

EQUILIBRIUM CHECK



$\sum F_z = 0$

$0.8kN - 2.8kN + 2kN = 0$

$\sum M_y^{①} = 0 \quad + \circlearrowleft$

$0.25kNm + 0.55kNm - 1kNm - 2.8kN \cdot 1m + 2kN \cdot 1.5m = 0$

ELEMENT SOLUTION

DEFLECTION

$\boxed{1}$

$$W_1(\xi_1) = \underset{1\times4}{\lfloor N \rfloor} \cdot \underset{4\times1}{\{q\}_1} = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} =$$

$$= N_3 \cdot q_3 + N_4 \cdot q_4 =$$

$$= \left(\frac{3}{l_1^2} \cdot \xi_1^2 - \frac{2}{l_1^3} \cdot \xi_1^3 \right) \cdot q_3 + \left(\frac{1}{l_1^2} \cdot \xi_1^3 - \frac{1}{l_1} \xi_1^2 \right) q_4$$

$$W_1(0) = 0 \quad , \quad W_1(l_1) = (3-2) q_3 + (1-1) q_4 = q_3$$

$$\frac{dW_1}{d\xi} = \left(\frac{6}{l_1^2} \xi_1 - \frac{6}{l_1^3} \xi_1^2 \right) q_3 + \left(\frac{3}{l_1^2} \xi_1^2 - \frac{2}{l_1} \xi_1 \right) q_4$$

$$\left. \frac{dW_1}{d\xi_1} \right|_{625\,mm} = 0 \implies W_{1\,min} = W_1(625) = -0.712\,mm$$

BENDING MOMENT

$$M_{y1}(\xi_1) = w_1'' EJ_{y1} = EJ_{y1} \cdot (N_3'' \cdot q_3 + N_4'' \cdot q_4) =$$

$$= EJ_{y1} \cdot \left(\left(\frac{6}{l_1^2} - \frac{12}{l_1^3} \xi_1 \right) \cdot q_3 + \left(\frac{6}{l_1^2} \xi_1 - \frac{2}{l_1} \right) q_4 \right)$$

$$M_{y1}(0) = -0.25 \cdot 10^6 \, Nmm$$

$$M_{y1}(l_1) = 0.55 \cdot 10^6 \, Nmm = M_1$$

SHEAR FORCE

$$T_{z1}(\xi_1) = -w_1''' EJ_{y1} = -EJ_{y1} (N_3''' \cdot q_3 + N_4''' \cdot q_4) =$$

$$= +EJ_{y1} \cdot \left(\frac{12}{l_1^3} q_3 - \frac{6}{l_1^2} q_4 \right) = -800 N$$

$\boxed{2}$

DEFLECTION

$$w_2(\xi_2) = \lfloor N \rfloor \cdot \{q\}_2 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} q_3 \\ q_5 \\ 0 \\ q_7 \end{Bmatrix} =$$

$$= N_1 \cdot q_3 + N_2 \cdot q_5 + N_4 \cdot q_7 =$$

$$= \left(1 - \frac{3}{l_2^2}\xi_2^2 + \frac{2}{l_2^3}\cdot\xi_2^3\right)q_3 + \left(\xi_2 - \frac{2}{l_2}\xi_2^2 + \frac{1}{l_2^2}\xi_2^3\right)q_5 + \left(\frac{1}{l_2^2}\xi_2^3 - \frac{1}{l_2}\xi_2^2\right)q_7$$

$$w_2(0) = q_3$$

$$w_2(l_2) = (1 - 3 + 2)q_3 + (l_2 - 2l_2 + l_2)q_5 + (l_2 - l_2)q_7 = 0$$

$$\frac{dw_2}{d\xi_2} = \left(\frac{6}{l_2^3}\xi_2^2 - \frac{6}{l_2^2}\xi_2\right)q_3 + \left(1 - \frac{4}{l_2}\xi_2 + \frac{3}{l_2^2}\xi_2^2\right)q_5 + \left(\frac{3}{l_2^2}\xi_2^2 - \frac{2}{l_2}\xi_2\right)q_7$$

$$\left.\frac{dw_2}{d\xi_2}\right|_{255.7mm} = 0 \quad \Rightarrow w_{2min} = w_2(255.7) = -0.3mm$$

BENDING MOMENT

$$M_{y2}(\xi_2) = w_2'' EJ_{y2} = EJ_{y2}\left(N_1''\cdot q_3 + N_2''\cdot q_5 + N_4''\cdot q_7\right) =$$

$$= EJ_{y2}\cdot\left(\left(\frac{12}{l_2^3}\xi_2 - \frac{6}{l_2^2}\right)q_3 + \left(\frac{6}{l_2^2}\xi_2 - \frac{4}{l_2}\right)q_5 + \left(\frac{6}{l_2^2}\xi_2 - \frac{2}{l_2}\right)q_7\right)$$

$$M_{y2}(0) = 1\cdot 10^6\ \text{Nmm} = M_2$$

$$M_{y2}(l_2) = 0$$

SHEAR FORCE

$$T_{Z2} = -w_2'''\cdot EJ_{y2} = -EJ_{y2}\cdot\left(N_1'''\cdot q_3 + N_2'''\cdot q_5 + N_4'''\cdot q_7\right) =$$

$$= -EJ_{y2}\cdot\left(\frac{12}{l_2^3}q_3 + \frac{6}{l_2^2}q_5 + \frac{6}{l_2^2}q_7\right) = 2000\ \text{N}$$

DEFORMATION AND INTERNAL FORCES